

$\mathbb{A}^1$ -EQUIVALENCE OF ZERO CYCLES ON SURFACES II

QIZHENG YIN AND YI ZHU

ABSTRACT. Using recent developments in the theory of mixed motives, we prove that the log Bloch conjecture holds for an open smooth complex surface if the Bloch conjecture holds for its compactification. This verifies the log Bloch conjecture for all  $\mathbb{Q}$ -homology planes and for open smooth surfaces which are not of log general type.

## 1. INTRODUCTION

Throughout this paper, we work with varieties over the complex numbers.

**1.1. Statement of the main theorem.** Let  $U$  be a smooth quasiprojective algebraic variety. Let

$$a : h_0(U)^0 \rightarrow \mathrm{Alb}(U)$$

be the Albanese morphism from the zeroth Suslin homology of degree zero to the Albanese variety of  $U$ , and let  $T(U) := \ker(a)$  be the Albanese kernel. When  $U$  is projective,  $h_0(U)$  reduces to the Chow group of zero cycles  $\mathrm{CH}_0(U)$ . Indeed, we get the classical Albanese map.

In dimension one, the Albanese morphism is well-understood by the classical work of Abel-Jacobi in the projective case, and by Rosenlicht in the open case.

**Theorem 1.1** (Abel-Jacobi, Rosenlicht [Ros52, Ros54]). *When  $\dim U = 1$ , the Albanese morphism is an isomorphism.*

The higher-dimensional analogue of Theorem 1.1 is much more subtle, although the torsion part of the Albanese morphism is known.

**Theorem 1.2** (Roĭtman [Roj80], Spieß-Szamuely [SS03]). *In arbitrary dimension, the Albanese morphism induces an isomorphism on torsion subgroups.*

In this paper, we study the two-dimensional case. In one direction, the log Mumford theorem says that the Albanese morphism fails to be injective as long as  $p_g(U) \neq 0$ .

---

*Date:* December 31, 2015.

*2010 Mathematics Subject Classification.* Primary 14C25, 14C15, 14F42.

*Key words and phrases.* Bloch's conjecture, open algebraic surfaces,  $\mathbb{Q}$ -homology plane, Suslin homology, mixed motives.

Q. Y. was supported by the grant ERC-2012-AdG-320368-MCSK.

**Theorem 1.3** (Mumford [Mum68], Zhu [Zhu15]). *Let  $U$  be a smooth algebraic surface with  $p_g(U) \neq 0$ . Then  $T(U)$  is infinite-dimensional.*

In the other direction, we expect the following conjecture. When  $U$  is projective, it is famously known as the Bloch conjecture [Blo80].

**Conjecture 1.4** (Log Bloch conjecture). *Let  $U$  be a smooth algebraic surface with  $p_g(U) = 0$ . Then*

$$T(U) = 0.$$

Using recent developments in the theory of mixed motives [ABV09, Ayo11, BVK14, Ayo15], we prove the following theorem.

**Theorem 1.5.** *Let  $(X, D)$  be a log smooth projective surface pair with interior  $U$ . If  $p_g(U) = 0$ , in particular,  $p_g(X) = 0$  as well, then the log Bloch conjecture holds for  $U$  if and only if it holds for  $X$ .*

Since the Bloch conjecture holds for any smooth projective surface  $X$  with  $\kappa(X) \leq 1$  [BKL76], our main theorem yields the following corollary.

**Corollary 1.6.** *The log Bloch conjecture holds for  $U$  if  $\kappa(X) \leq 1$ .  $\square$*

Since  $\kappa(X) \leq \kappa(U)$ , Corollary 1.6 generalizes the result of Bloch-Kas-Lieberman [BKL76] to open surfaces of  $\kappa(U) \leq 1$ . It also covers the second author's previous result [Zhu15] on the log Bloch conjecture for  $\kappa(U) = -\infty$ .

Further, we may apply Theorem 1.5 to the case where  $X$  is of general type and the Bloch conjecture is true. The Bloch conjecture holds in a great number of cases; see [BCP11, PW13, Voi14] for recent developments.

**1.2. Applications of Theorem 1.5 and Corollary 1.6.** The birational geometry of open surfaces is developed by Kawamata [Kaw79], while it is almost impossible to hope for a complete classification even for  $\kappa(U) \leq 1$ . We would like to focus on three special classes of surfaces whose geometry is extremely complicated.

**Example 1 ( $\kappa(U) = -\infty$ ): log del Pezzo surfaces**

Let  $U$  be the smooth locus of a singular del Pezzo surface of Picard number one with at worst quotient singularities. In general, such singular del Pezzo's form an unbounded family. Partial classifications are obtained in [KM99] with more than sixty exceptional collections. A difficult theorem of Keel-McKernan [KM99] states that  $U$  is log rationally connected. In particular, it implies the log Bloch conjecture for  $U$  [Zhu15, Prop. 4.3].

Since Theorem 1.5 and Corollary 1.6 do not depend on Keel-McKernan's result, we give a new proof of the following result.

**Corollary 1.7.** *With the notation as above, we have  $h_0(U) = \mathbb{Z}$ .*

**Example 2 ( $\kappa(U) = 0$ ): log Enriques surfaces**

A projective normal surface  $Y$  is said to be a *log Enriques surface* if

- (1)  $Y$  has at worst quotient singularities;

- (2)  $NK_Y \sim \mathcal{O}_Y$  for some positive integer  $N$ ;
- (3)  $\dim H^1(Y, \mathcal{O}_Y) = 0$ .

Since  $K_Y$  is  $\mathbb{Q}$ -Cartier, we define the *index*  $I$  of  $Y$  to be the smallest positive integer that  $IK_Y \sim \mathcal{O}_Y$ . By the work of Kawamata [Kaw79], Tsunoda [Tsu83], and Zhang [Zha91], the index is bounded by 66, while classically (when  $Y$  is smooth projective) it is bounded by 6.

**Corollary 1.8.** *Let  $U$  be the smooth locus of a log Enriques surface of index  $\geq 2$  defined as above. Then  $h_0(U) = \mathbb{Z}$ .*

Log Enriques surfaces are partially classified in [Zha91, Zha93, Kud02, Kud04]. There are more than 1000 examples of log Enriques surfaces with  $\delta$ -invariant 2 [Kud02].

*Proof of Corollaries 1.7, 1.8.* Let  $(X, D)$  be a minimal log resolution of  $U$ . By Corollary 1.6, the log Bloch conjecture holds in both cases. It suffices to show  $q(U) = 0$ . Since  $D$  is the exceptional set of the resolution of quotient singularities, we have  $q(U) = q(X)$ . Now the del Pezzo case follows from [Zha89, Lem. 1.1 (3)] and the Enriques case from [Zha91, Lem. 1.2].  $\square$

### Example 3: $\mathbb{Q}$ -homology planes

A smooth surface  $U$  is a  $\mathbb{Q}$ -homology plane if  $H^i(U, \mathbb{Q}) = H^i(\mathbb{A}^2, \mathbb{Q})$  for any  $i$ . A  $\mathbb{Q}$ -homology plane can have log Kodaira dimension  $-\infty, 0, 1$ , or  $2$ . Ramanujam [Ram71] constructed the first homology plane of log general type which is topologically contractible. They are classified for log Kodaira dimension  $\leq 1$ , but there is no thorough classification of  $\mathbb{Q}$ -homology planes of log general type [Miy01, Sect. 3.4].

Since all  $\mathbb{Q}$ -homology planes are rational [GP99], Corollary 1.6 implies:

**Corollary 1.9.** *Let  $U$  be a  $\mathbb{Q}$ -homology plane. Then the log Bloch conjecture holds, that is,  $h_0(U) = \mathbb{Z}$ .*  $\square$

The Bloch conjecture for fake projective planes remains unknown.

**1.3. Ideas from mixed motives.** The proof of our main theorem has two main ingredients. One is the work of Ayoub, Barbieri-Viale, and Kahn [ABV09, Ayo11, BVK14] on the derived category of 1-motives, especially the construction of a derived Albanese functor. The use is twofold: first, it gives a motivic interpretation of the Albanese morphism, allowing us to apply tools from the theory of mixed motives. Second, it provides a way to eliminate “easy” pieces of the motive of  $U$  (essentially 1-motives) while keeping track of the homological realization.

The other ingredient is the famous conservativity conjecture; see [Ayo15]. Regarded as one of the key conjectures in the study of motives, it notably says that a geometric motives is trivial if and only if its homological realization is trivial. By truncating the motive of  $U$  using the derived Albanese functor, we arrive at a motive which has trivial homological realization and whose motivic homology controls the Albanese kernel  $T(U)$ . Therefore, the

conservativity conjecture implies the log Bloch conjecture for  $U$ . Part of our main theorem then follows from a special case of the conservativity conjecture proven by Wildeshaus [Wil15].

Further, it is worth mentioning that the work of Bondarko-Sosnilo [BS14], if well-interpreted, might also lead to our results.

**1.4. Notation.** A *log pair*  $(X, D)$  means a variety  $X$  with a reduced Weil divisor  $D$ . We say that  $(X, D)$  is *log smooth* if  $X$  is smooth and  $D$  is a simple normal crossing divisor on  $X$ . A log pair is projective if the ambient variety is projective.

Given any smooth quasiprojective variety  $U$ , by the resolution of singularities, we may choose a log smooth projective compactification  $(X, D)$  with interior  $U$ . We use  $\kappa(X, D)$  to denote the *log Kodaira dimension*. We define the *log geometric genus*  $p_g(X, D) := \dim H^0(\Omega_X^{\dim X}(\log D))$  and the *log irregularity*  $q(X, D) := \dim H^0(\Omega_X^1(\log D))$ . Since they do not depend on the compactification, we may write  $\kappa(U)$ ,  $p_g(U)$ , and  $q(U)$  as well.

**Acknowledgment.** We would like to thank Joseph Ayoub for explaining his results. We thank Qile Chen and Javier Fresán for helpful discussions. This work was initiated during the AMS Summer Institute in Algebraic Geometry at the University of Utah, 2015. The authors would like to thank the Summer Institute for its hospitality and inspiring environment.

## 2. PRELIMINARIES

By Theorem 1.2, it suffices to consider the Albanese morphism with  $\mathbb{Q}$ -coefficients. From now on, all (co)homology, cycle groups, and motives are taken with  $\mathbb{Q}$ -coefficients.

**2.1. Mixed motives and conservativity.** We refer to [VSF00] for Voevodsky's theory of mixed motives. Since we work with  $\mathbb{Q}$ -coefficients, the categories of mixed motives in the Nisnevich and étale topologies are equivalent, with or without transfers; see [Ayo14].

Let  $\mathrm{DM}_{\mathrm{gm}}$  denote the triangulated category of geometric motives, and let  $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$  denote the triangulated category of effective geometric motives. We follow the homological convention. The unit object of  $\mathrm{DM}_{\mathrm{gm}}$  is denoted by  $\mathbb{Q}(0)$ , or simply  $\mathbb{Q}$ , and the Tate object  $\mathbb{Q}(1)$ . Given an object  $M \in \mathrm{DM}_{\mathrm{gm}}$ , its dual object  $\mathcal{H}om_{\mathrm{DM}_{\mathrm{gm}}}(M, \mathbb{Q})$  is denoted by  $M^\vee$ . The motive of a smooth variety  $Y$  is denoted by  $M(Y) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ .

The  $i$ -th motivic homology of  $M \in \mathrm{DM}_{\mathrm{gm}}$  is defined to be

$$h_i(M) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}[i], M).$$

For  $M = M(Y)$ , this recovers the  $i$ -th Suslin homology  $h_i(Y) = h_i(M(Y))$ .

Further, we refer to [Hub00] for the Hodge realization functor

$$R^H : \mathrm{DM}_{\mathrm{gm}} \rightarrow D^b(\mathrm{MHS}).$$

Composing with the forgetful functor  $D^b(\text{MHS}) \rightarrow D^b(\mathbb{Q})$ , we obtain the Betti realization  $R^B : \text{DM}_{\text{gm}} \rightarrow D^b(\mathbb{Q})$ . Recall the statement of the conservativity conjecture.

**Conjecture 2.1** (see [Ayo15, Conj. 2.1]). *The Betti realization functor  $R^B$  is conservative. In other words, a morphism  $f : M \rightarrow N$  in  $\text{DM}_{\text{gm}}$  is an isomorphism if and only if  $R^B(f) : R^B(M) \rightarrow R^B(N)$  is an isomorphism.*

Using consequences of the standard conjecture D for abelian varieties [AK02], Kimura-O’Sullivan finiteness [Kim05], and Bondarko’s weight structures [Bon09, Bon10], Wildeshaus proved the following special case of the conservativity conjecture.

**Theorem 2.2** (Wildeshaus [Wil15, Th. 1.12]). *Let  $\text{DM}_{\text{gm}}^{\text{ab}} \subset \text{DM}_{\text{gm}}$  denote the smallest triangulated subcategory containing the motives of smooth curves and closed under direct summands, tensor products, and duality. Then the restriction of  $R^B$  to  $\text{DM}_{\text{gm}}^{\text{ab}}$  is conservative.*

By introducing  $\text{DM}_{\text{gm}}^{\text{ab}}$ , we may reformulate our main theorem as follows.

**Theorem 2.3.** *Under the assumption as in Theorem 1.5, the following three conditions are equivalent:*

- (1)  $T(U) = 0$ ;
- (2)  $T(X) = 0$ ;
- (3)  $M(U), M(X) \in \text{DM}_{\text{gm}}^{\text{ab}}$ .

**2.2. Derived category of 1-motives.** We shall mainly follow the book of Barbieri-Viale-Kahn [BVK14]. Let  $\mathcal{M}_1$  denote Deligne’s category of 1-motives [Del74] with  $\mathbb{Q}$ -coefficients. By [Org04, Th. 3.4.1], the bounded derived category  $D^b(\mathcal{M}_1)$  can be naturally identified with the thick triangulated subcategory of  $\text{DM}_{\text{gm}}^{\text{eff}}$  generated by the motives of smooth curves, denoted by  $d_{\leq 1} \text{DM}_{\text{gm}}^{\text{eff}}$ . The identification is compatible with realizations [Vol12]. For simplicity we always make this identification.

One of the main results of [BVK14] is the construction of a *derived Albanese functor*.

**Theorem 2.4** ([BVK14, Cor. 6.2.2]). *The inclusion  $d_{\leq 1} \text{DM}_{\text{gm}}^{\text{eff}} \hookrightarrow \text{DM}_{\text{gm}}^{\text{eff}}$  admits a left adjoint*

$$L \text{ Alb} : \text{DM}_{\text{gm}}^{\text{eff}} \rightarrow d_{\leq 1} \text{DM}_{\text{gm}}^{\text{eff}}.$$

We list a number of results and facts about the functor  $L \text{ Alb}$ , which will be used in the proof of our main theorem. To begin with, when  $Y$  is a smooth variety, we write  $L \text{ Alb}(Y) = L \text{ Alb}(M(Y))$ . Then the natural morphism  $M(Y) \rightarrow L \text{ Alb}(Y)$  induces a morphism in motivic homology

$$(2.1) \quad h_0(Y) \rightarrow h_0(L \text{ Alb}(Y)).$$

By [BVK14, Lem. 13.4.2], we have

$$h_0(L \text{ Alb}(Y))^0 = \text{Alb}(Y) \otimes \mathbb{Q},$$

and the degree zero part of (2.1) coincides with the Albanese morphism.

The next statement concerns the Hodge realization of  $L \operatorname{Alb}(M)$  for  $M \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{eff}}$ . Recall that a mixed Hodge structure  $H$  is *effective* if the  $(i, j)$ -th part of the weight-graded piece  $\operatorname{Gr}_{i+j}^W H$  vanishes unless  $i, j \leq 0$ . Given an effective mixed Hodge structure  $H$ , let  $H_{\leq 1}$  denote the maximal quotient of  $H$  of weights  $\geq -2$  and of types  $(0, 0)$ ,  $(0, -1)$ ,  $(-1, 0)$ , and  $(-1, -1)$ .

**Theorem 2.5** ([BVK14, Th. 15.3.1]). *For  $M \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{eff}}$ , the morphism  $M \rightarrow L \operatorname{Alb}(M)$  induces isomorphisms*

$$H_i(R^H(M))_{\leq 1} \xrightarrow{\sim} H_i(R^H(L \operatorname{Alb}(M))).$$

The theorem above applies to  $L \operatorname{Alb}(Y)$  and also to the Borel-Moore variant of  $L \operatorname{Alb}(Y)$ . Let  $M^c(Y) \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{eff}}$  denote the motive of  $Y$  with compact support. By [VSF00, Ch. 5, Th. 4.3.7], there is an isomorphism

$$M^c(Y) \simeq M(Y)^\vee(\dim Y)[2 \dim Y].$$

We write  $L \operatorname{Alb}^c(Y) = L \operatorname{Alb}(M^c(Y))$ .

**Corollary 2.6** ([BVK14, Cor. 15.3.2]). *By Theorem 2.5, we have*

$$H_i(R^H(L \operatorname{Alb}(Y))) = \begin{cases} H_0(Y, \mathbb{Q}) & i = 0 \\ H_1(Y, \mathbb{Q}) & i = 1 \\ H_2(Y, \mathbb{Q})_{\leq 1} & i = 2 \\ 0 & i < 0 \text{ or } i > 2 \end{cases}$$

and

$$H_i(R^H(L \operatorname{Alb}^c(Y))) = \begin{cases} H_0^{\operatorname{BM}}(Y, \mathbb{Q}) & i = 0 \\ H_1^{\operatorname{BM}}(Y, \mathbb{Q}) & i = 1 \\ H_i^{\operatorname{BM}}(Y, \mathbb{Q})_{\leq 1} & 2 \leq i \leq \dim Y + 1 \\ 0 & i < 0 \text{ or } i > \dim Y + 1. \end{cases}$$

Finally, we recall the fact that  $\mathcal{M}_1$  is of cohomological dimension one [Org04, Prop. 3.2.4]. Hence, all elements in  $D^b(\mathcal{M}_1)$  can be represented by complexes with zero differentials. In particular, we have

$$L \operatorname{Alb}(Y) \simeq \bigoplus_{i=0}^2 L_i \operatorname{Alb}(Y)[i] \quad \text{and} \quad L \operatorname{Alb}^c(Y) \simeq \bigoplus_{i=0}^{\dim Y + 1} L_i \operatorname{Alb}^c(Y)[i],$$

with  $L_i \operatorname{Alb}(Y), L_i \operatorname{Alb}^c(Y) \in \mathcal{M}_1$ ; see [BVK14, Cor. 9.2.3, Prop. 10.6.2]. When  $\dim Y = 1$ , this gives the ‘‘Chow-K nneth’’ decomposition of  $M(Y)$  [BVK14, Cor 11.1.1]

$$(2.2) \quad M(Y) \simeq L \operatorname{Alb}(Y) \simeq \bigoplus_{i=0}^2 L_i \operatorname{Alb}(Y)[i].$$

### 3. PROOF OF THE MAIN THEOREM

In this section we prove our main theorem, that is, Theorem 2.3.

**3.1. Proof of (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3).** For (1)  $\Rightarrow$  (2), consider a partial compactification  $U \subset Y \subset X$  such that  $C = Y \setminus U$  is a smooth curve. By induction, it suffices to show that  $T(U) = 0$  implies  $T(Y) = 0$ .

Recall the Gysin distinguished triangle [VSF00, Ch. 5, Prop. 3.5.4]

$$M(U) \rightarrow M(Y) \rightarrow M(C)(1)[2] \rightarrow M(U)[1].$$

By applying the functor  $L \operatorname{Alb}$ , we find a morphism of distinguished triangles

$$(3.1) \quad \begin{array}{ccccccc} M(U) & \longrightarrow & M(Y) & \longrightarrow & M(C)(1)[2] & \longrightarrow & M(U)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L \operatorname{Alb}(U) & \longrightarrow & L \operatorname{Alb}(Y) & \longrightarrow & \mathbb{Q}(1)[2] & \longrightarrow & L \operatorname{Alb}(U)[1]. \end{array}$$

Here we used the fact that  $L \operatorname{Alb}(M(C)(1)) \simeq \mathbb{Q}(1)$  [BVK14, Prop. 8.2.3]. Moreover, the morphism

$$M(C)(1) \rightarrow L \operatorname{Alb}(M(C)(1)) \simeq \mathbb{Q}(1)$$

coincides with the projection in (2.2)

$$M(C) \rightarrow L_0 \operatorname{Alb}(C) \simeq \mathbb{Q}$$

twisted by  $\mathbb{Q}(1)$ .

Now we apply motivic homology to the distinguished triangles in (3.1). Since  $h_0(U) \rightarrow h_0(Y)$  is surjective [Zhu15, Lem. 4.2] and

$$h_0(\mathbb{Q}(1)[2]) = \operatorname{CH}_{-1}(\operatorname{pt}) = 0,$$

we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} h_1(M(C)(1)[2]) & \longrightarrow & h_0(U) & \longrightarrow & h_0(Y) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ h_1(\mathbb{Q}(1)[2]) & \longrightarrow & h_0(L \operatorname{Alb}(U)) & \longrightarrow & h_0(L \operatorname{Alb}(Y)) & \longrightarrow & 0. \end{array}$$

The first vertical arrow is surjective since it comes from a projection. The middle vertical arrows are given by the Albanese morphisms of  $U$  and  $Y$ . Our assumption  $T(U) = 0$  says that the second vertical arrow is injective. Then, by the five lemma, the third vertical arrow is also injective, and hence  $T(Y) = 0$ .

For (2)  $\Rightarrow$  (3), a result of Guletskiĭ-Pedrini [GP03, Th. 7] shows that  $T(X) = 0$  if and only if  $M(X) \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$ . By applying several Gysin triangles, we also know that  $M(X) \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$  if and only if  $M(U) \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$ .  $\square$

**3.2. Proof of (3)  $\Rightarrow$  (1).** Consider the distinguished triangle

$$(3.2) \quad M'(U) \rightarrow M(U) \rightarrow L \operatorname{Alb}(U) \rightarrow M'(U)[1].$$

Our assumption  $p_g(U) = 0$  says that  $H_2(U, \mathbb{Q}) = H_2(U, \mathbb{Q})_{\leq 1}$ . Then, by Theorem 2.5 and Corollary 2.6, we have

$$H_i \left( R^B(M'(U)) \right) = \begin{cases} H_3(U, \mathbb{Q}) & i = 3 \\ H_4(U, \mathbb{Q}) & i = 4 \\ 0 & i < 3 \text{ or } i > 4. \end{cases}$$

Next, consider the motive  $M'(U)^\vee(2)[4]$ , whose Betti realization is

$$H_i \left( R^B(M'(U)^\vee(2)[4]) \right) = \begin{cases} H_0^{\operatorname{BM}}(U, \mathbb{Q}) & i = 0 \\ H_1^{\operatorname{BM}}(U, \mathbb{Q}) & i = 1 \\ 0 & i < 0 \text{ or } i > 1. \end{cases}$$

It fits in a distinguished triangle

$$L \operatorname{Alb}(U)^\vee(2)[4] \rightarrow M^c(U) \rightarrow M'(U)^\vee(2)[4] \rightarrow L \operatorname{Alb}(U)^\vee(2)[5].$$

Since  $L \operatorname{Alb}(U)^\vee(2)[4] \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{eff}}$  by Cartier duality [BVK14, Prop. 4.5.1], we have  $M'(U)^\vee(2)[4] \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{eff}}$ . This allows us to apply the functor  $L \operatorname{Alb}$  to  $M'(U)^\vee(2)[4]$ . By Theorem 2.5 and Corollary 2.6, the morphism

$$(3.3) \quad M'(U)^\vee(2)[4] \rightarrow L \operatorname{Alb}(M'(U)^\vee(2)[4])$$

induces an isomorphism

$$R^B(M'(U)^\vee(2)[4]) \xrightarrow{\sim} R^B(L \operatorname{Alb}(M'(U)^\vee(2)[4])).$$

We are ready to apply conservativity. Our assumption  $M(U) \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$  implies  $M^c(U) \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$ . Moreover, since  $d_{\leq 1} \operatorname{DM}_{\operatorname{gm}}^{\operatorname{eff}} \subset \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$ , we have  $L \operatorname{Alb}(U)^\vee(2)[4] \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$  and hence  $M'(U)^\vee(2)[4] \in \operatorname{DM}_{\operatorname{gm}}^{\operatorname{ab}}$ . Then, according to Theorem 2.2, the morphism (3.3) is itself an isomorphism.

We thus obtain from (3.2) a distinguished triangle

$$(3.4) \quad L \operatorname{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4] \rightarrow M(U) \rightarrow L \operatorname{Alb}(U) \\ \rightarrow L \operatorname{Alb}(M'(U)^\vee(2)[4])^\vee(2)[5].$$

Taking motivic homology, we have an exact sequence

$$h_0 \left( L \operatorname{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4] \right) \rightarrow h_0(U) \rightarrow h_0(L \operatorname{Alb}(U)),$$

where the second arrow is given by the Albanese morphism of  $U$ . Hence, to prove  $T(U) = 0$ , it suffices to show that

$$h_0 \left( L \operatorname{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4] \right) = 0.$$

For this we observe that

$$L \operatorname{Alb}(M'(U)^\vee(2)[4]) \simeq \bigoplus_{i=0}^1 L_i \operatorname{Alb}(M'(U)^\vee(2)[4])[i] \simeq \bigoplus_{i=0}^1 L_i \operatorname{Alb}^c(U)[i].$$



Here we have used the fact that the Hodge realization gives a full embedding  $\mathcal{M}_1 \subset \text{MHS}$  [Del74, Sect. 10.1.3]. We compute

$$\begin{aligned}
 & h_0 \left( L \text{Alb} \left( M'(U)^\vee(2)[4] \right)^\vee(2)[4] \right) \\
 &= h_0 \left( \bigoplus_{i=0}^1 (L_i \text{Alb}^c(U)[i])^\vee(2)[4] \right) \\
 &= \text{Hom}_{\text{DM}_{\text{gm}}} \left( \mathbb{Q}, \bigoplus_{i=0}^1 (L_i \text{Alb}^c(U)[i])^\vee(2)[4] \right) \\
 &= \text{Hom}_{\text{DM}_{\text{gm}}} \left( \bigoplus_{i=0}^1 L_i \text{Alb}^c(U)[i], \mathbb{Q}(2)[4] \right) \\
 &= \text{Hom}_{\text{DM}_{\text{gm}}} (L_0 \text{Alb}^c(U), \mathbb{Q}(2)[4]) \oplus \text{Hom}_{\text{DM}_{\text{gm}}} (L_1 \text{Alb}^c(U), \mathbb{Q}(2)[3]).
 \end{aligned}$$

By [BVK14, Prop. 10.6.2], we have

$$L_0 \text{Alb}^c(U) \simeq \begin{cases} \mathbb{Q} & \text{if } U \text{ is projective} \\ 0 & \text{if not.} \end{cases}$$

Since

$$\text{Hom}_{\text{DM}_{\text{gm}}} (\mathbb{Q}, \mathbb{Q}(2)[4]) = \text{CH}_{-2}(\text{pt}) = 0,$$

we find in both cases  $\text{Hom}_{\text{DM}_{\text{gm}}} (L_0 \text{Alb}^c(U), \mathbb{Q}(2)[4]) = 0$ .

Further, by [BVK14, Cor. 12.11.2], the 1-motive  $L_1 \text{Alb}^c(U)$  is represented by a two-term complex in degrees 0 and  $-1$

$$\mathbb{Q}^{\oplus r} \rightarrow A \otimes \mathbb{Q},$$

where  $A$  is an abelian variety and  $r = \#\{\text{connected components of } D\} - 1$ . In other words, there is an extension of 1-motives

$$(3.5) \quad 0 \rightarrow (A \otimes \mathbb{Q})[-1] \rightarrow L_1 \text{Alb}^c(U) \rightarrow \mathbb{Q}^{\oplus r} \rightarrow 0,$$

which yields an exact sequence

$$\begin{aligned}
 \text{Hom}_{\text{DM}_{\text{gm}}} (\mathbb{Q}, \mathbb{Q}(2)[3])^{\oplus r} &\rightarrow \text{Hom}_{\text{DM}_{\text{gm}}} (L_1 \text{Alb}^c(U), \mathbb{Q}(2)[3]) \\
 &\rightarrow \text{Hom}_{\text{DM}_{\text{gm}}} ((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]).
 \end{aligned}$$

Since

$$\text{Hom}_{\text{DM}_{\text{gm}}} (\mathbb{Q}, \mathbb{Q}(2)[3]) = \text{CH}_{-2}(\text{pt}, 1) = 0,$$

it suffices to show that  $\text{Hom}_{\text{DM}_{\text{gm}}} ((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]) = 0$ .

We may assume  $A$  to be the Albanese variety of a smooth projective surface  $S$  (which exists by the Lefschetz hyperplane theorem). Recall the Chow-Künneth decomposition of  $M(S)$  [Mur90, Th. 3]

$$M(S) \simeq \bigoplus_{i=0}^4 M_i(S)[i].$$

We have  $M_{4-i}(S) \simeq M_i(S)^\vee(2)$  and  $M_1(S) \simeq (A \otimes \mathbb{Q})[-1]$ . Hence

$$\begin{aligned}
 (3.6) \quad \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(S), \mathbb{Q}(2)[3]) \\
 &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, M_3(S)[3]) \\
 &= \mathrm{CH}_0(M_3(S)[3]) \\
 &= 0,
 \end{aligned}$$

where the last equality follows again from [Mur90, Th. 3]. The proof of Theorem 2.3 is now complete.  $\square$

**3.3. “Chow-Künneth” decomposition.** Our proof of Theorem 2.3 also leads to the following consequence.

**Corollary 3.1.** *Assume one of the equivalent conditions in Theorem 2.3. Then  $M(U)$  admits a “Chow-Künneth” decomposition*

$$M(U) \simeq \bigoplus_{i=0}^2 L_i \mathrm{Alb}(U)[i] \oplus \bigoplus_{i=3}^4 L_{4-i} \mathrm{Alb}^c(U)^\vee(2)[i].$$

*In particular, it is Kimura-O’Sullivan finite.*

*Proof.* Assuming  $M(U) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{ab}}$ , we have obtained in (3.4) a distinguished triangle

$$\begin{aligned}
 \bigoplus_{i=3}^4 L_{4-i} \mathrm{Alb}^c(U)^\vee(2)[i] &\rightarrow M(U) \rightarrow \bigoplus_{i=0}^2 L_i \mathrm{Alb}(U)[i] \\
 &\rightarrow \bigoplus_{i=3}^4 L_{4-i} \mathrm{Alb}^c(U)^\vee(2)[i+1].
 \end{aligned}$$

For the distinguished triangle to split, it suffices to show that

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\left(\bigoplus_{i=0}^2 L_i \mathrm{Alb}(U)[i], \bigoplus_{i=3}^4 (L_{4-i} \mathrm{Alb}^c(U))^\vee(2)[i+1]\right) = 0.$$

The left-hand side consists of six direct summands, all of which can be computed explicitly. To keep the paper short we shall only do the most complicated one, that is,

$$(3.7) \quad \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1 \mathrm{Alb}(U)[1], L_1 \mathrm{Alb}^c(U)^\vee(2)[4]).$$

By [BVK14, Cor. 9.2.3], the 1-motive  $L_1 \mathrm{Alb}(U)$  is represented by the two-term complex in degrees 0 and  $-1$

$$0 \rightarrow \mathrm{Alb}(U) \otimes \mathbb{Q}.$$

This gives an extension of 1-motives

$$(3.8) \quad 0 \rightarrow (\mathbb{G}_m \otimes \mathbb{Q})^{\oplus s}[-1] \rightarrow L_1 \mathrm{Alb}(U) \rightarrow (A' \otimes \mathbb{Q})[-1] \rightarrow 0,$$

where  $A'$  is the abelian part of the semi-abelian variety  $\mathrm{Alb}(U)$ . Again we assume  $A'$  to be the Albanese variety of a smooth projective surface  $S'$ , and hence  $(A' \otimes \mathbb{Q})[-1] \simeq M_1(S')$ . We also have  $(\mathbb{G}_m \otimes \mathbb{Q})[-1] \simeq \mathbb{Q}(1)$ .

Combining (3.5) and (3.8), we see that (3.7) sits in the middle of several extensions involving the following four terms:

- (1)  $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}} (M_1(S')[1], M_3(S)[4]);$
- (2)  $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}} (M_1(S')[1], \mathbb{Q}(2)[4]);$
- (3)  $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}} (\mathbb{Q}(1)[1], M_3(S)[4]);$
- (4)  $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}} (\mathbb{Q}(1)[1], \mathbb{Q}(2)[4]).$

The vanishing of the second term is shown in (3.6) (with  $S'$  replaced by  $S$ ). The vanishing of the three other terms follows from the fact that given two Chow motives  $M$  and  $M'$ , we have  $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}} (M, M'[i]) = 0$  for all  $i > 0$  [VSF00, Ch. 5, Cor. 4.2.6]. Hence (3.7) vanishes.

Finally, by [Maz04, Rem. 5.11], all elements in  $d_{\leq 1} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$  are Kimura-O'Sullivan finite. The last statement follows since Kimura-O'Sullivan finiteness is closed under direct sums and tensor products.  $\square$

On the other hand, there exist motives of smooth surfaces which are not Kimura-O'Sullivan finite [Maz04, Th. 5.18].

## REFERENCES

- [ABV09] Joseph Ayoub and Luca Barbieri-Viale. 1-motivic sheaves and the Albanese functor. *J. Pure Appl. Algebra*, 213(5):809–839, 2009.
- [AK02] Yves André and Bruno Kahn. Nilpotence, radicaux et structures monoïdales. *Rend. Sem. Mat. Univ. Padova*, 108:107–291, 2002. With an appendix by Peter O'Sullivan.
- [Ayo11] Joseph Ayoub. The  $n$ -motivic  $t$ -structures for  $n = 0, 1$  and  $2$ . *Adv. Math.*, 226(1):111–138, 2011.
- [Ayo14] Joseph Ayoub. A guide to (étale) motivic sheaves. *Preprint*, 2014. To appear in *Proceedings of ICM 2014*.
- [Ayo15] Joseph Ayoub. Motives and algebraic cycles: a selection of conjectures and open questions. *Preprint*, 2015.
- [BCP11] Ingrid Bauer, Fabrizio Catanese, and Roberto Pignatelli. Surfaces of general type with geometric genus zero: a survey. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 1–48. Springer, Heidelberg, 2011.
- [BKL76] S. Bloch, A. Kas, and D. Lieberman. Zero cycles on surfaces with  $p_g = 0$ . *Compositio Math.*, 33(2):135–145, 1976.
- [Blo80] Spencer Bloch. *Lectures on algebraic cycles*. Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., 1980.
- [Bon09] M. V. Bondarko. Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky versus Hanamura. *J. Inst. Math. Jussieu*, 8(1):39–97, 2009.
- [Bon10] M. V. Bondarko. Weight structures vs.  $t$ -structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-Theory*, 6(3):387–504, 2010.
- [BS14] M. V. Bondarko and V. A. Sosnilo. Detecting the  $c$ -effectivity of motives, their weights, and dimension via Chow-weight (co)homology: a “mixed motivic decomposition of the diagonal”. *Preprint*, 2014. arXiv:1411.6354.
- [BVK14] Luca Barbieri-Viale and Bruno Kahn. On the derived category of 1-motives. *Preprint*, 2014. To appear in *Astérisque*.
- [Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

- [GP99] R. V. Gurjar and C. R. Pradeep.  $\mathbf{Q}$ -homology planes are rational. III. *Osaka J. Math.*, 36(2):259–335, 1999.
- [GP03] V. Guletskiĭ and C. Pedrini. Finite-dimensional motives and the conjectures of Beilinson and Murre. *K-Theory*, 30(3):243–263, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part III.
- [Hub00] Annette Huber. Realization of Voevodsky’s motives. *J. Algebraic Geom.*, 9(4):755–799, 2000.
- [Kaw79] Yujiro Kawamata. On the classification of noncomplete algebraic surfaces. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 215–232. Springer, Berlin, 1979.
- [Kim05] Shun-Ichi Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.
- [KM99] Seán Keel and James McKernan. Rational curves on quasi-projective surfaces. *Mem. Amer. Math. Soc.*, 140(669):viii+153, 1999.
- [Kud02] S. A. Kudryavtsev. Classification of logarithmic Enriques surfaces with  $\delta = 2$ . *Mat. Zametki*, 72(5):715–722, 2002.
- [Kud04] S. A. Kudryavtsev. Classification of Enriques log surfaces with  $\delta = 1$ . *Mat. Zametki*, 76(1):87–96, 2004.
- [Maz04] Carlo Mazza. Schur functors and motives. *K-Theory*, 33(2):89–106, 2004.
- [Miy01] Masayoshi Miyanishi. *Open algebraic surfaces*, volume 12 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2001.
- [Mum68] D. Mumford. Rational equivalence of 0-cycles on surfaces. *J. Math. Kyoto Univ.*, 9:195–204, 1968.
- [Mur90] J. P. Murre. On the motive of an algebraic surface. *J. Reine Angew. Math.*, 409:190–204, 1990.
- [Org04] Fabrice Orgogozo. Isomotifs de dimension inférieure ou égale à un. *Manuscripta Math.*, 115(3):339–360, 2004.
- [PW13] Claudio Pedrini and Charles Weibel. Some surfaces of general type for which Bloch’s conjecture holds. *Preprint*, 2013. To appear in *Period domains, algebraic cycles, and arithmetic*, Cambridge Univ. Press.
- [Ram71] C. P. Ramanujam. A topological characterisation of the affine plane as an algebraic variety. *Ann. of Math. (2)*, 94:69–88, 1971.
- [Roj80] A. A. Rojzman. The torsion of the group of 0-cycles modulo rational equivalence. *Ann. of Math. (2)*, 111(3):553–569, 1980.
- [Ros52] Maxwell Rosenlicht. Equivalence relations on algebraic curves. *Ann. of Math. (2)*, 56:169–191, 1952.
- [Ros54] Maxwell Rosenlicht. Generalized Jacobian varieties. *Ann. of Math. (2)*, 59:505–530, 1954.
- [SS03] Michael Spieß and Tamás Szamuely. On the Albanese map for smooth quasi-projective varieties. *Math. Ann.*, 325(1):1–17, 2003.
- [Tsu83] Shuichiro Tsunoda. Structure of open algebraic surfaces. I. *J. Math. Kyoto Univ.*, 23(1):95–125, 1983.
- [Voi14] Claire Voisin. Bloch’s conjecture for Catanese and Barlow surfaces. *J. Differential Geom.*, 97(1):149–175, 2014.
- [Vol12] Vadim Vologodsky. Hodge realizations of 1-motives and the derived Albanese. *J. K-Theory*, 10(2):371–412, 2012.
- [VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.
- [Wil15] Jörg Wildeshaus. On the interior motive of certain Shimura varieties: the case of Picard surfaces. *Manuscripta Math.*, 148(3-4):351–377, 2015.

- [Zha89] De-Qi Zhang. Logarithmic del Pezzo surfaces with rational double and triple singular points. *Tohoku Math. J. (2)*, 41(3):399–452, 1989.
- [Zha91] De-Qi Zhang. Logarithmic Enriques surfaces. *J. Math. Kyoto Univ.*, 31(2):419–466, 1991.
- [Zha93] De-Qi Zhang. Logarithmic Enriques surfaces. II. *J. Math. Kyoto Univ.*, 33(2):357–397, 1993.
- [Zhu15] Yi Zhu.  $\mathbb{A}^1$ -equivalence of zero cycles on surfaces. *Preprint*, 2015. arXiv: 1510.01712.

(Yin) DEPARTEMENT MATHEMATIK, ETH ZÜRICH, RÄMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND

*E-mail address:* qizheng.yin@math.ethz.ch

(Zhu) PURE MATHEMATICS, UNIVERISTY OF WATERLOO, WATERLOO, ON N2L3G1, CANADA

*E-mail address:* yi.zhu@uwaterloo.ca